2701. Differentiating implicitly with respect to y,

$$\frac{dx}{dy}y^3 + 3xy^2 - 2\frac{dx}{dy} = 0.$$

Setting $\frac{dx}{dy} = 0$ for tangents parallel to the y axis,

$$\begin{aligned} &3xy^2 = 0 \\ \implies &x = 0 \text{ or } y = 0. \end{aligned}$$

Substituting $x = 0$ in gives $0 = 3$, so no values. Substituting $y = 0$ gives $-2x = 3$, so $x = -3/2$. The curve is therefore parallel to the y axis at $(-3/2, 0)$.

2702. This is true if the alternative hypothesis has the form $H_1 : \mu > \mu_0$, i.e. if testing for a possible *increase* in the population mean.

If, however, the alternative hypothesis is in the opposite direction, that is $H_1 : \mu < \mu_0$, then the statement should finish "$\mathbb{P}(\bar{X} - \mu_0 < c) = 0.05$".

2703. Assume, for a contradiction, that $y = \text{f}(x)$ is cubic and convex everywhere. Hence, $\text{f}''(x) > 0$ for all $x \in \mathbb{R}$. Also, $\text{f}''(x)$ is the second derivative of a cubic, so it is linear, of the form

$$\text{f}''(x) = ax + b.$$

The only way for a linear function to be positive everywhere is if $a = 0$. But, integrating twice, this makes $\text{f}(x)$ a quadratic, which is a contradiction. Hence, no cubic graph is convex everywhere. □

2704. There are ${}^9\text{C}_2 = 36$ ways of shading two squares. Now, consider the ways of doing so, such that they *do* share a border. With the two squares adjacent horizontally, there are $2 \times 3 = 6$ ways of doing this, so there are 12 ways of doing so overall. This gives $36 - 12 = 24$ ways of shading two squares such that they do not share a border.

2705. The parabolae are tangent. Hence, the equation $\text{f}(x) - \text{g}(x) = 0$ for intersections has a double root. And, since it is a quadratic equation, it can have no other roots/factors. So, it can be expressed as

$$\text{f}(x) - \text{g}(x) = a(x - b)^2,$$

for some constants a and b. Since a square is non-negative, this cannot change sign. QED.

2706. Consecutive odd numbers differ by 2. Calling them n and $n + 2$,

$$\begin{aligned} &\frac{1}{n} + \frac{1}{n+2} = \frac{84}{1763} \\ \implies &1763\big((n+2) + n\big) = 84n(n+2) \\ \implies &-84n^2 + 3358n + 3526 = 0 \\ \implies &n = -\tfrac{43}{42}, 41. \end{aligned}$$

Rejecting the non-integer root, the numbers are 41 and 43.

2707. At $\theta = \frac{\pi}{6}$, $V_{\text{actual}} = \sqrt{3}/2$. The approximation is

$$\begin{aligned} V_{\text{approx}} &= 1 - \tfrac{1}{2}\left(\tfrac{\pi}{6}\right)^2 + \tfrac{1}{24}\left(\tfrac{\pi}{6}\right)^4 \\ &= 0.86605388... \end{aligned}$$

So, the percentage error is

$$\begin{aligned} \frac{V_{\text{approx}} - V_{\text{actual}}}{V_{\text{actual}}} &= \frac{0.866... - \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} \\ &\approx 0.0033\%. \end{aligned}$$

2708. We use the (secondary) log rule $\log_a b \equiv \log_{a^n} b^n$. Writing each logarithm over base 8,

$$\begin{aligned} &\log_8 2y = \log_2 x + \log_{16} x \\ \implies &\log_8 2y = \log_8 x^3 + \log_8 x^{\frac{3}{4}} \\ \implies &\log_8 2y = \log_8 x^{\frac{15}{4}} \\ \implies &2y = x^{\frac{15}{4}} \\ \implies &y = \tfrac{1}{2}x^{\frac{15}{4}}. \end{aligned}$$

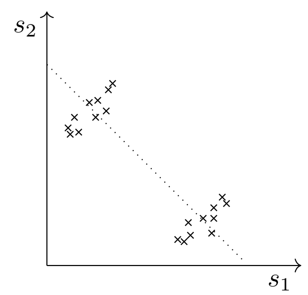
2709. (a) For an object of mass m_2 at the surface of the Earth, NII gives the acceleration as

$$\begin{aligned} a &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times m_2}{6370000^2 \times m_2} \\ &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6370000^2} \\ &\approx 9.81 \text{ ms}^{-2}. \end{aligned}$$

(b) Increasing r, the distance to Earth's centre, to 6380 km, the acceleration is

$$\begin{aligned} a &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times m_2}{6380000^2 \times m_2} \\ &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6380000^2} \\ &\approx 9.78 \text{ ms}^{-2}. \end{aligned}$$

2710. This is possible if the individual sets form distinct subpopulations, as shown below.



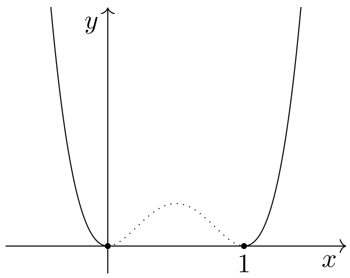
Each subpopulation is positively correlated, but the subpopulations themselves are grouped along a line with negative gradient. So, the combined correlation is negative.

2711. Squaring both sides and factorising,

$$y = x^2(x-1)^2.$$

This is a quartic with double roots at $x = 0$ and $x = 1$. This is sketched below (solid and dotted sections together).

But we only get points on the *original* curve if $x^2 - x \geq 0$, which is over the domain $\mathbb{R} \setminus (0, 1)$. So, the curve, now ignoring the dotted section, is



2712. The area is given by

$$A = \tfrac{1}{2}r^2\theta.$$

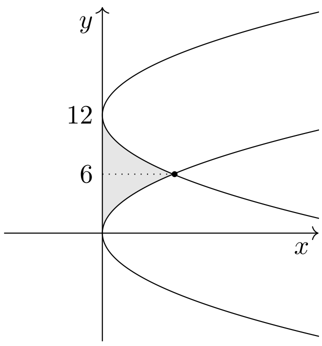
Differentiating both sides of this with respect to time, using the product and chain rules,

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}\left(\tfrac{1}{2}r^2\right)\cdot\theta + \tfrac{1}{2}r^2\cdot\frac{d}{dt}(\theta)\\ &= r\frac{dr}{dt}\theta + \tfrac{1}{2}r^2\frac{d\theta}{dt}.\end{aligned}$$

We can then replace $r\theta$ in the first term with the standard arc length formula $l = r\theta$. This gives the required result:

$$\frac{dA}{dt} = l\frac{dr}{dt} + \tfrac{1}{2}r^2\frac{d\theta}{dt}.$$

2713. (a) The two parabolae are tangent to the y axis at $y = 0$ and $y = 12$, and intersect at $y = 6$:



(b) By splitting along the dotting line above, we can express the A as twice a single integral:

$$\begin{aligned}A &= 2\int_0^6 y^2\,dy\\ &= 2\left[\tfrac{1}{3}y^3\right]_0^6\\ &= 2\times\tfrac{1}{3}\times 6^3\\ &= 144, \text{ as required.}\end{aligned}$$

2714. Differentiating implicitly with the product rule,

$$\begin{aligned}&y(\ln x + 1) = e^x\\ \Longrightarrow\ &\frac{dy}{dx}(\ln x + 1) + y\left(\tfrac{1}{x}\right) = e^x.\end{aligned}$$

Setting $\frac{dy}{dx} = 0$ for SPs,

$$y\left(\tfrac{1}{x}\right) = e^x \implies y = xe^x.$$

Substituting this back into the original equation, $xe^x(\ln x + 1) = e^x$. Since e^x cannot be zero, we can divide by it, giving $x\ln x + x - 1 = 0$. This can be solved numerically by e.g. Newton-Raphson:

$$x_{n+1} = x_n - \frac{x_n \ln x_n + x_n - 1}{\ln x_n + 2},$$

starting with $x_n = 0.5$. The iteration converges to $x = 1$. So, there is a stationary point at $(1, e)$.

2715. The binomial expansion gives

$$\left(\sqrt{x} \pm x\right)^4 \equiv x^2 \pm 4x^{\frac{5}{2}} + 6x^3 \pm 4x^{\frac{7}{2}} + x^4.$$

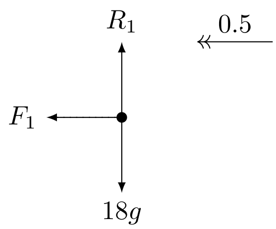
Adding the $\pm$ versions, the terms with fractional indices cancel, leaving the original equation as

$$\begin{aligned}&2x^2 + 12x^3 + 2x^4 = 4 + 12x^3\\ \Longrightarrow\ &x^4 + x^2 - 2 = 0\\ \Longrightarrow\ &(x^2+2)(x^2-1) = 0.\end{aligned}$$

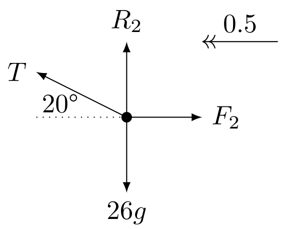
The first factor yields no real roots, so $x = \pm 1$.

2716. The sum of the interior angles is 3π radians, so the mean of the AP, which is the third term, must be $\theta_3 = \frac{3}{5}\pi$. This is closer to 0 than to the upper limit of 2π. So, a smallest angle of $\theta_1 = 0$ is the relevant bound, which makes $\theta_2 = {}^{3\pi}/_{10}$. The lower bound is not attainable, but the upper bound is. So, the set of possible values is $\left({}^{3\pi}/_{10}, {}^{3\pi}/_{5}\right]$.

2717. (a) For the load:



For the combined load and sledge:



(b) The system is moving, so F_2 is at maximal friction. Resolving horizontally and vertically,

$$\begin{aligned} T\cos 20° - 0.1R_2 &= 26 \times 0.5, \\ T\sin 20° + R_2 - 26g &= 0. \end{aligned}$$

Adding ten times the first to the second,

$$\begin{aligned} 10T\cos 20° + T\sin 20° &= 384.8 \\ \implies T &= \frac{348.8}{10\cos 20° + \sin 20°} \\ &= 31.8150... \end{aligned}$$

So, the tension is 31.8 N (3sf).

(c) Horizontally for the load, $F_1 = 18 \times 0.5 = 9$. And $R_1 = 18g$. So, the combined contact force (reaction and friction) has magnitude

$$\begin{aligned} C &= \sqrt{(18g)^2 + 9^2} \\ &= 176.629... \\ &= 176 \text{ N (3sf)}. \end{aligned}$$

2718. Solving simultaneously for intersections, we sub for x^2. This gives

$$\begin{aligned} 4y - y^2 &= 4 \\ \implies (y-2)^2 &= 0. \end{aligned}$$

Since the root at $y = 2$ is a double root, this must be a point of tangency, as required.

2719. We can set aside the $+1$ in the integrand, which will simply add 4 to the final result, and consider

$$\int_0^4 \frac{1}{8-\sqrt{x}}\, dx$$

Let $u = 8-\sqrt{x}$, so $\sqrt{x} = 8-u$. Also $\frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$, which gives $dx = -2(8-u)\, du$.

$$\begin{aligned} &\int_0^4 \frac{1}{8-\sqrt{x}}\, dx \\ &= \int_8^6 \tfrac{1}{u} \cdot -2(8-u)\, du \\ &= 2\int_6^8 \tfrac{8}{u} - 1\, du. \end{aligned}$$

Carrying out the integration, this is

$$\begin{aligned} &2\Big[8\ln|u| - u\Big]_6^8 \\ &= 2(8\ln 8 - 8) - 2(8\ln 6 - 6) \\ &= 16\ln\tfrac{8}{6} - 4 \\ &= 16\ln\tfrac{4}{3} - 4. \end{aligned}$$

Reinstating the original $+1$ adds 4 to the result, giving $16\ln\frac{4}{3}$ as required.

2720. The probability of picking one of each is

$$2 \times \frac{n}{100} \times \frac{100-n}{99} \equiv \frac{n(100-n)}{4950}.$$

Setting this equal to $\frac{16}{33}$,

$$\begin{aligned} \frac{n(100-n)}{4950} &= \frac{16}{33} \\ \implies n &= 40, 60. \end{aligned}$$

2721. (a) Differentiating implicitly,

$$\begin{aligned} & x^2 - 3xy + y^2 = 1 \\ \implies & 2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \\ \implies & \frac{dy}{dx}(2y - 3x) = 3y - 2x \\ \implies & \frac{dy}{dx} = \frac{3y-2x}{2y-3x}. \end{aligned}$$

(b) The gradient of the proposed normal is $-\frac{3}{7}$. So, the gradient of the curve at the point of intersection should be $\frac{7}{3}$. This gives

$$\begin{aligned} & \frac{3y-2x}{2y-3x} = \frac{7}{3} \\ \implies & 9y - 6x = 14y - 21x \\ \implies & y = 3x. \end{aligned}$$

Solving simultaneously with $x^2 - 3xy + y^2 = 1$, we get $x = \pm 1$. Checking these points, the line $3x + 7y = 24$ is normal to the curve at $(1, 3)$.

2722. We want the angle at the origin in a triangle with vertices at $(0,0,0)$, $(1,1,0)$, $(0,1,1)$. Calculating the side lengths, we get $\sqrt{2}$ for each. Hence, the triangle is equilateral, and the angle is 60°.

2723. (a) The average on s_2 was higher, mainly due to the lower end of the range being much higher for s_2 than for s_1. The spread was lower for s_2 than for s_1, for the same reason.

(b) There is a weak/moderate positive correlation between the marks on the two assessments.

(c) The ten highest-attaining students may be identified by placing a line $s_1 + s_2 = k$, and increasing k until only 10 students are above and to the right of it. Having done this, we see a strong negative correlation between the higher scores.

(d) Despite the overall +ve correlation, the result in (c) is not very surprising: defining a group by high total score $s_1 + s_2$ gives that group a boundary with a gradient of -1. This makes negative correlation likely.

2724. Let $u = x + 1$ and $v = x^2 + x + 1$. The quotient rule gives

$$\begin{aligned} y &= \frac{u}{v} \\ \implies \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\ &= -\frac{x^2 + 2x}{(x^2 + x + 1)^2}. \end{aligned}$$

Setting the numerator to zero for SPs, $x = 0, -2$. To differentiate again by the quotient rule, let $p = x^2 + 2x$ and $q = (x^2 + x + 1)^2$:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{p}{q} \\ \implies \frac{d^2y}{dx^2} &= -\frac{p'q - pq'}{q^2} \\ &= \frac{-(2x+2)(x^2+x+1)^2 + (x^2+2x)\cdot 2(x^2+x+1)(2x+1)}{(x^2+x+1)^4} \\ &\equiv \frac{2x^3 + 6x^2 - 2}{(x^2+x+1)^3}. \end{aligned}$$

Evaluating this at $x = 0$ gives -2, and at $x = -2$ gives $2/9$. Hence, there is a local maximum at $(0, 1)$ and a local minimum at $(-2, -1/3)$.

2725. The equation $f(x) = 0$ has 1 root and $g(x) = 0$ has 2 roots. All three are distinct. Hence, the graphs have as many roots as numerator zeros, and as many asymptotes as denominator zeros. In (c), the square changes the nature of the asymptote (double rather than single), but doesn't change the number of asymptotes.

(a) 1 root, 2 asymptotes.

(b) 2 roots, 1 asymptote.

(c) 1 root, 2 asymptotes.

2726. Using a double-angle formula, we can rewrite the denominator as

$$1 + \cos 2x \equiv 2\cos^2 x.$$

This gives a standard integral:

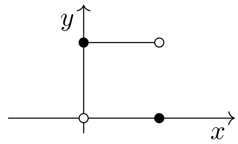
$$\begin{aligned} &\int \frac{2}{1 + \cos 2x}\, dx \\ =& \int \sec^2 x\, dx \\ =&\ \tan x + c. \end{aligned}$$

2727. Multiplying up by the denominator of the LHS, we require

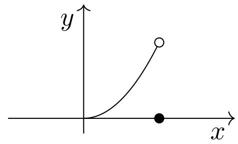
$$x^2 + x - 1 \equiv A(x^2 + 1) + Bx^2.$$

Equating the coefficients of x, we get $1 = 0$. Hence, it is not possible to form such an identity.

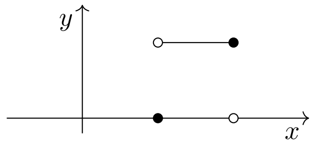
2728. (a) The graph $y = S(x)$ is a step, zero everywhere apart from on $[0, 1)$.



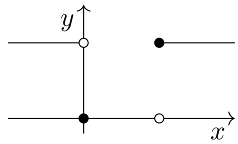
(b) The graph $y = x^2\,S(x)$ is $y = x^2$ on the interval $[0, 1)$, and zero everywhere else:



(c) The graph $y = S(2 - x)$ is a reflection of $y = S(x)$ in the line $x = 1$.



(d) Applying the function once gives outputs of 0 and 1. Applying the function again, as in $y = S(S(x))$, switches these over:



2729. Enacting the differential operator,

$$\begin{aligned} &\frac{d}{dx}(\sin x + \cos y) = 1 \\ \implies& \cos x - \sin y \frac{dy}{dx} = 1 \\ \implies& \sin y \frac{dy}{dx} = \cos x - 1 \\ \implies& \frac{dy}{dx} = \frac{\cos x - 1}{\sin y}, \text{ as required.} \end{aligned}$$

2730. (a) Translating into algebra,

$$\begin{aligned} &\frac{y^2}{x} = -k(x - 2) \\ \implies& y^2 = -k(x^2 - 2x) \\ \implies& y^2 + k(x^2 - 2x) = 0 \\ \implies& k(x-1)^2 + y^2 = k. \end{aligned}$$

(b) Since $k > 0$, this is an ellipse, centred on $(1, 0)$.

2731. The LHS is the sum of an AP, with first term 1 and common difference 2. Hence, quoting the standard formula, the equation is

$$\begin{aligned} &\tfrac{1}{2}n(2 + 2(n-1)) = 100 \\ \implies& n^2 = 100. \end{aligned}$$

Assuming $n \geq 1$, this gives $n = 10$.

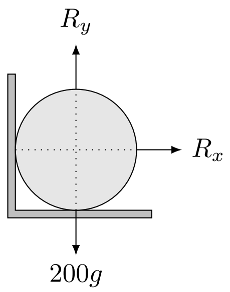
2732. At points of inflection, the second derivative is zero. So, $12x^2 - 2p = 0$. Substituting in $x = \pm 1$, we get $p = 6$.

— Nota Bene —

For a stationary point, the second derivative must be zero and also change sign. But the direction of implication in this question doesn't require us to consider this. All we need to know is that, **if** there are points of inflection, **then** the second derivative must be zero.

2733. (a) Consider moments around the centre (axis of symmetry) of the bale. The reaction forces from the shovel are radial, so they pass through the centre and can have no moment around it. Only the frictional forces at the points of contact produce a moment. They are at the same distance from the centre, so, since the bale is not rotating, the magnitudes of the forces must be equal.

(b) Assume the frictional forces to be negligible. The reaction forces act perpendicular to the shovel, as below:



Horizontal $F = ma$ is $R_x = 200 \times 1 = 200$ N. Vertical $F = ma$ is $R_y - 200g = 200 \times 2$, which gives $R_y = 2360$ N.

2734. Differentiating by the chain rule, $\frac{dx}{du} = 2\sin u \cos u$. This is $\sin 2u$, by a double-angle formula. Taking the reciprocal,

$$\frac{du}{dx} = \frac{1}{\sin 2u} = \operatorname{cosec} 2u, \text{ as required.}$$

2735. This is true.

- The critical region is at the upper tail (from $H_1 : p > 1/4$), so comparing $\mathbb{P}(X \geq a)$ with $1/50 = 2\%$ is the right comparison to make.
- If the p-value (the probability of a result as extreme as the one in question) is greater than the significance level, then the result isn't all that strange. In formal language, there is insufficient evidence to reject H_0.
- The acceptance region is the complement of the critical region.

2736. Both top and bottom have $x = 1$ as a root, which means we can't yet take the limit. But we can divide top and bottom by $(x - 1)$:

$$\begin{aligned} &\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1} \\ &= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} \\ &= \lim_{x\to 1} \frac{x+1}{x^2+x+1} \\ &= \tfrac{2}{3}, \text{ as required.} \end{aligned}$$

2737. (a) True,
(b) True,
(c) False: the result should be ${}^4\text{C}_2 = 6$.

2738. Solving by separation of variables,

$$\begin{aligned} &\frac{dy}{dx} - 2y = 0 \\ \implies &\frac{dy}{dx} = 2y \\ \implies &\int \frac{1}{y}\, dy = \int 2\, dx \\ \implies &\ln|y| = 2x + c \\ \implies &|y| = e^{2x+c}. \end{aligned}$$

Redefining the constant of integration, the general solution of the DE is $y = Ae^{2x}$.

— Nota Bene —

The formal steps in the redefinition are as follows:

> Integrating both sides, we reach
>
> $$\begin{aligned} &|y| = e^{2x+c} \\ \implies &|y| = e^c \times e^{2x}. \end{aligned}$$
>
> Both $|y|$ and e^c are necessarily positive. But y can thereby be negative. We offer this option by writing $y = Ae^{2x}$, where A can now be any constant.

This extra justification is overkill at A-level. It is common practice to leave out the mod signs entirely in a DE solution, and simply to write $\ln y = 2x + c$, leading to $y = e^{2x+c}$ and then to $y = Ae^{2x}$.

2739. F is a quadratic. Completing the square,

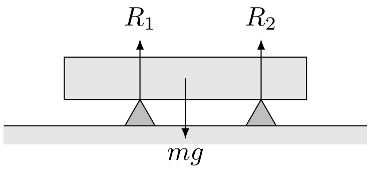
$$\text{F}(x) = (x+3)^2 - 8.$$

So, the range of F is $[-8, \infty)$. G is a quadratic in x^2. Again completing the square,

$$\text{G}(x) = \left(x^2 - \tfrac{1}{2}p\right)^2 - p - \tfrac{1}{4}p^2.$$

So, for the same range, we require $-p - \frac{1}{4}p^2 = -8$. This gives $p = 4, -8$. But, for $p < 0$, the squared bracket cannot equal zero, so the quartic cannot attain -8. Hence, the only value is $p = 4$.

2740. Without loss of generality, let p, q, r be lengths in metres. The force diagram is



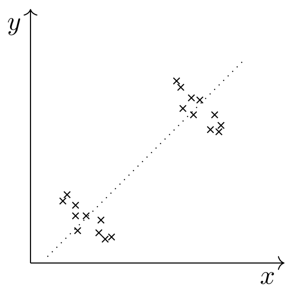
The centre of mass is $\frac{1}{2}(p+q+r)-p$ from the LH support, which simplifies to $\frac{1}{2}(-p+q+r)$. The equations are

$$\overset{\frown}{\text{LH}} : \tfrac{1}{2}(-p+q+r)mg - qR_2 = 0$$
$$\updownarrow : R_1 + R_2 = mg.$$

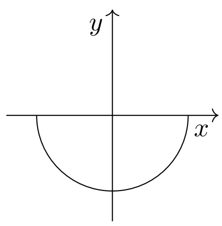
Rearranging the former, we get $2R_2 = \frac{-p+q+r}{q}mg$. Subtracting this from the latter gives

$$\begin{aligned} R_1 - R_2 &= mg - \frac{-p+q+r}{q}mg \\ &\equiv \frac{q-(-p+q+r)}{q}mg \\ &\equiv \frac{p-r}{q}mg, \text{ as required.} \end{aligned}$$

2741. If the individual sample means $(\bar{x}, \bar{y})$ were equal, then it would be true. But the following, in which the sample means differ significantly relative to the spreads of the samples, is a counterexample:



2742. Let $\arcsin x = \theta$. Then $x = \sin\theta$ and $y = -\cos\theta$, for $\theta \in [-\pi/2, \pi/2]$. This gives a semicircle:



2743. Call the two numbers a and ϕa, with $a \neq 0$ and $\phi > 1$. Then we have

$$\begin{aligned} \frac{a\phi}{a} &= \frac{a + a\phi}{a\phi} \\ \Longrightarrow \phi^2 - \phi - 1 &= 0 \\ \Longrightarrow \phi &= \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

Since $\phi > 1$, we take the positive root.

2744. Differentiating by the quotient rule,

$$\begin{aligned} y &= \frac{2}{1+x^2} - 1 \\ \Longrightarrow \frac{dy}{dx} &= \frac{-4x}{\left(1+x^2\right)^2} \\ \Longrightarrow \frac{d^2y}{dx^2} &= \frac{-4\left(1+x^2\right)^2 + 4x \cdot 4x\left(1+x^2\right)}{\left(1+x^2\right)^4} \\ &\equiv \frac{-4+12x^2}{\left(1+x^2\right)^3}. \end{aligned}$$

The second derivative is zero at

$$\begin{aligned} -4 + 12x^2 &= 0 \\ \Longrightarrow x &= \pm 1/\sqrt{3}. \end{aligned}$$

Furthermore, each of these is a single root of the numerator, so the second derivative changes sign. Hence, these are points of inflection, as required.

——— NOTA BENE ———

Broadly, there are three methods for showing a sign change in the second derivative for a point of inflection:

① Multiplicity of roots, as above.
② Analysing the behaviour of the function, e.g. knowing that $\sin x$ changes sign at its roots.
③ Showing that the third derivative is non-zero.

2745. (a) To get r successes, there must be $n-r$ failures. The probability of this, in one particular order (one branch of the tree diagram), is $p^r q^{n-r}$. We multiply this by the number of branches, which is the number of orders. This is the number of ways of choosing r locations for the successes, which is ${}^n\mathrm{C}_r$.

(b) Quoting the formula

$${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!},$$

this result is simply the addition of all possible outcomes of the binomial distribution. These must add to 1 by definition.

2746. Let $u = 2\ln t + 1$ and $v' = t$, so $u' = \frac{2}{t}$ and $v = \frac{1}{2}t^2$. Applying the parts formula,

$$\begin{aligned} &\int_0^1 t(2\ln t + 1)\,dt \\ &= \left[\tfrac{1}{2}t^2(2\ln t + 1)\right]_0^1 - \int t\,dt \\ &= \left[t^2 \ln t\right]_0^1 \\ &= \ln 1 \\ &= 0, \text{ as required.} \end{aligned}$$

2747. (a) For small positive t, the quadratic factor $t^2 - t$ is $-$ve, while e^{-t} is positive. So, the rate of production is $-$ve: the process absorbs gas.

(b) Setting $t(t-1)e^{-t} = 0$, we get $t = 0$ or $t = 1$. Between these times, the rate is negative; after $t = 1$, the rate is positive.

(c) Differentiating by the product rule,

$$\begin{aligned}\frac{d^2V}{dt^2} &= (2t-1)e^{-t} - (t^2 - t)e^{-t} \\ &\equiv -(t^2 - 3t + 1)e^{-t}.\end{aligned}$$

The rate is stationary at $t^2 - 3t + 1 = 0$. The positive root is $t \approx 2.6$ s.

(d) We integrate the rate to find the total:

$$V = \int_0^{t_{\text{end}}} (t^2 - t)e^{-t}\, dt.$$

The process is terminated at some large value of t_{end}. Using the definite integration facility on a calculator, with such a large value, the total volume is 1 cubic metre.

— Nota Bene —

It's the command word "find", as opposed to "determine", that allows use of a calculator here. If the instruction is to "find", then find the answer by any means! If the instruction had been "determine", then parts would have been the technique.

2748. Writing the top in terms of the bottom,

$$2x^2 - x - 7 \equiv 2(x+2)^2 - 9(x+2) - 6.$$

Hence, the curve is

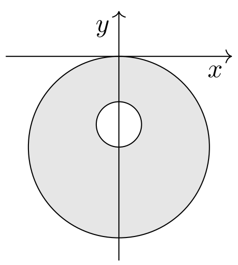
$$y = 2(x+2) - 9 - \frac{6}{x+2}.$$

As $x \to \infty$ the last term tends to zero, leaving the asymptote $y = 2(x+2) - 9 \equiv 2x - 5$.

2749. The boundary equations are circles.

① The first is centred at $(0, -4)$, radius 4.
② The second is centred at $(0, -3)$, radius 1.

Hence, the second circle lies entirely within the first circle.



The region is the area within the larger circle, but outside the smaller circle. Since the circles do not intersect, we can subtract: $4\pi - 1\pi = 3\pi$.

2750. The base is coloured red with probability $\frac{1}{3}$. If it is red, then no other faces can be red; this outcome has probability $\frac{1}{3} \cdot \left(\frac{2}{3}\right)^4 = \frac{16}{243}$.

The base is not red with probability $\frac{2}{3}$. If it is not red, then a number of options remain for the four remaining faces. By number of red faces:

(0) Success,
(1) Success,
(2) Success for outcomes RNRN and NRNR,
(3) Failure,
(4) Failure.

This gives

$$\frac{2}{3} \cdot \left(\left(\tfrac{2}{3}\right)^4 + 4\left(\tfrac{2}{3}\right)^3 \tfrac{1}{3} + 2\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^2\right) = \frac{112}{243}.$$

So, the overall probability is $\frac{16}{243} + \frac{112}{243} = \frac{128}{243}$.

— Alternative Method —

The possibility space consists of $3^5 = 243$ equally likely outcomes. If these, successful outcomes have no two red faces sharing a border. Classifying these by the number of red faces:

(0) Each face can be one of two colours, giving $2^5 = 32$ outcomes.
(1) There are five choices for the red face, then 2^4 colourings for the rest. This gives $5 \times 2^4 = 80$ outcomes.
(2) The base cannot be red. Then there are two choices for the (opposite) locations of the red faces, followed by 2^2 options for the other sloped faces: $2 \times 2 \times 2^2 = 16$ outcomes.

Overall, the probability is

$$p = \frac{32 + 80 + 16}{243} = \frac{128}{243}.$$

2751. (a) A cubic has rotational symmetry around its point of inflection. And, in this case, the two outer roots are equidistant from the central one, meaning they are rotationally symmetrical around it. Hence, the central root must be at the point of inflection.

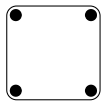
(b) At a point of inflection, the second derivative is zero. So, we require $6x - 42 = 0$, which gives $x = 7$. This must be a root, so $p = 143$.

(c) The cubic is monic. Hence, according to the factor theorem, the three roots must multiply to give 315. Since $315 \div 7 = 45$, the solution is $x = 5, 7, 9$.

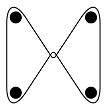
2752. If two forces of a Newton III pair act on the same object, then, because they are equal in magnitude and opposite in direction, they must sum to zero when calculating both the resultant force and the resultant moment on that object. And, since the object is rigid, neither can they have any effect on its shape.

— Nota Bene —

If we were to set aside the "rigid" requirement, a counterexample would be a loop of extensible string passed around four pegs. If no NIII pair is exerted, then the string forms a square:



If, however, the string is twisted, then the string exerts a Newton III pair on itself. This pair has no translational or rotational effect on the object (the loop of string), but it does have an effect: it stops the string returning to a square.



2753. The range of $\cos^2 x$ is $[0,1]$. This is affected by output transformations, but unaffected by input transformations. This gives the ranges as

(a) $[0,1]$,
(b) $[0,a]$,
(c) $[b,b+a]$.

2754. We exponentiate each side, over base a:

$$a^{\text{LHS}} = a^{\log_a b} \equiv b.$$

Also,

$$a^{\text{RHS}} = a^{\frac{\log_c b}{\log_c a}} \equiv a^{\log_c b \log_c x} \equiv c^{\log_c b} \equiv b.$$

This proves the result.

— Alternative Method —

Consider $a^k = b$. Taking logs base c,

$$\begin{aligned} \log_c a^k &= \log_c b \\ \Longrightarrow k\log_c a &= \log_c b \\ \Longrightarrow k &= \frac{\log_c b}{\log_c a}. \end{aligned}$$

We also know that $k = \log_a b$. So,

$$\log_a b \equiv \frac{\log_c b}{\log_c a}, \text{ as required.}$$

2755. (a) Using the quotient rule,

$$\begin{aligned} y &= \frac{x^2+x}{x-1} \\ \Longrightarrow \frac{dy}{dx} &= \frac{(2x+1)(x-1)-(x^2+x)}{(x-1)^2} \\ &\equiv \frac{x^2-2x-1}{(x-1)^2}. \end{aligned}$$

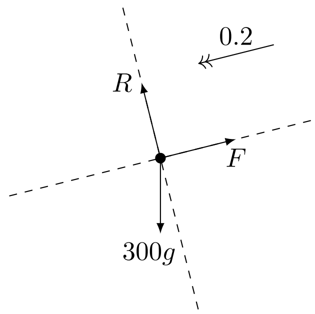
Setting the numerator to zero for SPs, we get $x^2-2x-1=0 \implies x = 1 \pm \sqrt{2}$.

(b) Using the quotient rule again, the second derivative is

$$\begin{aligned} &\frac{(2x-2)(x-1)^2-(x^2-2x-1)2(x-1)}{(x-1)^4} \\ \equiv\ &\frac{4}{(x-1)^3}. \end{aligned}$$

Evaluating at $x = 1\pm\sqrt{2}$, the second derivative is $\pm\sqrt{2}$. Hence, $x = 1+\sqrt{2}$ is a local minimum, and $x = 1-\sqrt{2}$ is a local maximum.

2756. (a) The force diagram for the pallet is



Resolving down the slope,

$$\begin{aligned} 300g\sin 8° - F &= 300 \times 0.2 \\ \Longrightarrow F &= 349.16... \end{aligned}$$

Resolving perpendicular to the slope,

$$\begin{aligned} R - 300g\cos 8° &= 0 \\ \Longrightarrow R &= 2911.3... \end{aligned}$$

The contact force is the resultant of these two, which is $\sqrt{F^2+R^2} = 2930$ N (3sf).

— Nota Bene —

Why did we choose to put the frictional force forwards in the above diagram?

If the van floor were smooth, then, on an 8° slope, the acceleration of the pallet would be $g\sin 8° = 1.36...$ ms^{-2} down the slope. In fact, the pallet is accelerating down the slope at 0.2 ms^{-2}. Since $0.2 < 1.36...$, the friction must be acting forwards.

(b) The angle above the slope is $\arctan\frac{R}{F}$, which is 83.16°. Above the forwards horizontal, then, the angle is $83.16° + 8° = 91.2°$ (1dp). As an acute angle, this is 88.8° above the backwards horizontal.

2757. Yes, it does imply that h is convex. We are told that the gradient increases as the input increases. And, since h is a polynomial function, it can't have any asymptotes, discontinuities or other strange behaviours. An increasing gradient means that the second derivative (rate of change of gradient) is positive. This is the definition of "convex".

2758. To be well defined, each element of the domain must produce exactly one element of the codomain as its image. To be invertible, the mapping must be one-to-one.

(a) Well defined, but not invertible.
(b) Well defined, and invertible.
(c) Not well defined.
(d) Well defined, but not invertible.
(e) Well defined, and invertible.
(f) Not well defined.

2759. We know that

$$f'(x) = -\frac{x-\mu}{\sigma^2}\, f(x).$$

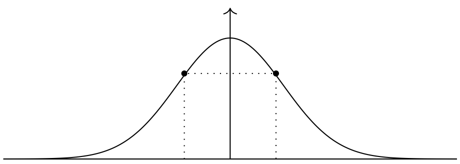
Differentiating again by the product rule and then substituting the original result for $f'(x)$ back in,

$$\begin{aligned} f''(x) &= \left(-\frac{x-\mu}{\sigma^2}\right)' f(x) - \frac{x-\mu}{\sigma^2}\, f'(x) \\ &= -\frac{1}{\sigma^2}\, f(x) + \frac{(x-\mu)^2}{\sigma^4}\, f(x) \\ &\equiv \frac{-\sigma^2 + (x-\mu)^2}{\sigma^4}\, f(x). \end{aligned}$$

At $x = \mu \pm \sigma$, the numerator of the fraction is $-\sigma^2 + (\pm\sigma)^2$, which is zero. The RHS is zero, so the LHS $f''(x) = 0$, as required.

— Nota Bene —

This result is the fact that the points of inflection of the normal bell curve are one standard deviation either side of the mean:



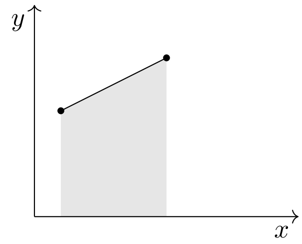
2760. Assume, for a contradiction, that there exists a non-constant polynomial function f for which $f(x) = f(x+1)$ for all $x \in \mathbb{R}$.

Consider the function g defined over $\mathbb{R}$ by

$$g(x) = f(x) - f(0).$$

This is also a non-constant polynomial function. Furthermore, $g(0) = 0$. Hence, $g(1) = 0$, $g(2) = 0$ and so on. This gives $g(z) = 0$ for all integers z. So, the polynomial equation $g(z) = 0$ has infinitely many roots. This is a contradiction. So, there is no polynomial function f for which $f(x) = f(x+1)$ for all $x \in \mathbb{R}$. □

2761. (a) The endpoints of the line segment, at $t = 0, 2$, have coordinates $(1, 4)$ and $(5, 6)$. This gives



(b) $\frac{dx}{dt} = 2$, which gives $A = \int_0^2 2t + 8\, dt = 20.$

(c) The trapezium has $A = \frac{1}{2} \times 4 \times (4 + 6) = 20.$

2762. (a) The particles begin with the same horizontal position. Hence, to collide, they must have the same horizontal component of velocity u_x:

$$u \cos\theta_1 = u \cos\theta_2.$$

Since $u \neq 0$, this has solution $\theta_1 = \pm\theta_2$. The particles start at different vertical positions, so we require $\theta_1 = -\theta_2$.

— Nota Bene —

Algebraically, there are other possibilities such as $\theta_1 = 2\pi + \theta_2$. But physically we can assume that the angles of projection are acute.

(b) Vertically, the positions are

$$\begin{aligned} y_1 &= d - u\sin\theta t - \tfrac{1}{2}gt^2, \\ y_2 &= u\sin\theta t - \tfrac{1}{2}gt^2. \end{aligned}$$

Equating these, the terms in t^2 cancel:

$$\begin{aligned} d - u\sin\theta t &= u\sin\theta t \\ \Longrightarrow t &= \frac{d}{2u\sin\theta}. \end{aligned}$$

So, the horizontal position is

$$\begin{aligned} x &= u\cos\theta t \\ &= u\cos\theta \times \frac{d}{2u\sin\theta} \\ &\equiv \tfrac{1}{2}d\cot\theta, \text{ as required.} \end{aligned}$$

2763. (a) Multiplying up, we get $y(x+y) - 1 = 0$, which is $y^2 + xy - 1 = 0$.

i. Rearranging, $x = \dfrac{1-y^2}{y}$.
ii. This is a quadratic in y. The formula gives

$$y = \frac{-x \pm \sqrt{x^2+4}}{2}.$$

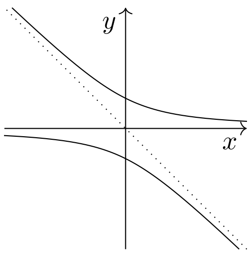
(b) i. With x as the subject, there is division by y. Hence, we have an asymptote at $y = 0$.
ii. With y as the subject, the negative root tends to $y = \frac{-x-x}{2} = -x$ as $x \to \infty$.

(c) Differentiating the equation from (a) i,

$$\frac{dx}{dy} = -y^{-2} - 1.$$

For a tangent parallel to the y axis, we require $y^{-2} + 1 = 0$. But this has no roots, because y^2 is non-negative.

(d) Joining the dots, the curve is



2764. The possibility space is

	1	2	3	4	5	6
1						✓
2						✓
3						✓
4						✓
5						✓
6	✓	✓	✓	✓	✓	✓

Restricting the possibility space,

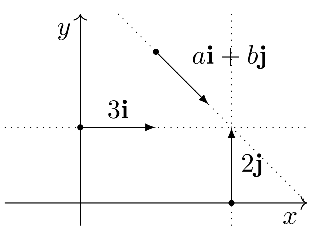
$$\mathbb{P}(\text{at least one 6} \mid \text{sum of 8}) = \tfrac{9}{15} = \tfrac{3}{5}.$$

2765. Differentiating twice by chain and quotient rules,

$$\begin{aligned} \mathrm{f}(x) &= \ln(1 + e^x) \\ \implies \mathrm{f}'(x) &= \frac{e^x}{1 + e^x} \\ \implies \mathrm{f}''(x) &= \frac{e^x(1 + e^x) - e^{2x}}{(1 + e^x)^2} \\ &\equiv \frac{e^x}{(1 + e^x)^2}. \end{aligned}$$

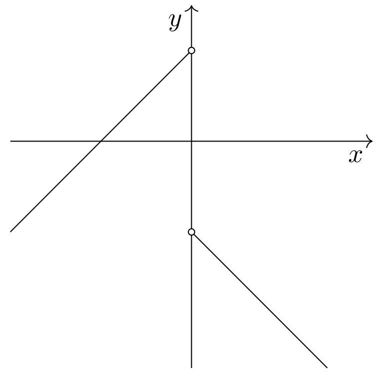
Since exponentials are positive, both numerator and denominator are positive for all x, hence the function f is convex everywhere, as required.

2766. Assume, for a contradiction, that the object is in equilibrium. Then the lines of action of the three forces must be concurrent, as below.



Hence, $a = -b$. But equilibrium also needs $a = 2$ and $b = -3$. This is a contradiction. Hence, the object cannot remain in equilibrium.

2767. The factor $\frac{|x|}{x}$ is a step function: it has value $+1$ for positive x and -1 for negative x. Hence, we sketch $y = 4 + x$ for positive x and $y = -(4 + x)$ for negative x. At $x = 0$ the graph is undefined.



2768. (a) $\ln(x^2) \equiv 2 \ln x$,

(b) $\ln(e^2 \cdot x) \equiv \ln e^2 + \ln x \equiv 2 + \ln x$,

(c) $\ln(e^2 \div x) \equiv \ln e^2 - \ln x \equiv 2 - \ln x$.

2769. For SPs, the derivative is zero:

$$\begin{aligned} & 4x^3 - 12x^2 - 40x = 0 \\ \implies & x = -2, 0, 5. \end{aligned}$$

The curve is a positive quartic, so $x = -2$ and $x = 5$ are local minima. These have coordinates $(-2, k - 32)$ and $(5, k - 375)$. The latter is lower. So, for the curve to have no roots, we require $k \in (375, \infty)$.

2770. The negations of these statements are the two parts of the factor theorem: "$(x - \alpha)$ is a factor" and "α is a root". The implication in the factor theorem is $\iff$, so the implication is also two-way here.

2771. We integrate by inspection, spotting that $2x$ is the derivative of the inside function $x^2 + 4$.

$$\int 2x\sqrt{x^2 + 4}\, dx \equiv \tfrac{2}{3}(x^2 + 4)^{\frac{3}{2}} + c.$$

— Alternative Method —

Let $u = x^2 + 4$. Then $\frac{du}{dx} = 2x$, so $du = 2x\, dx$. Rearranging the integrand and then enacting the substitution,

$$\begin{aligned} & \int 2x\sqrt{x^2 + 4}\, dx \\ = & \int \sqrt{x^2 + 4} \cdot 2x\, dx \\ = & \int \sqrt{u}\, du \\ = & \tfrac{2}{3}u^{\frac{3}{2}} + c. \end{aligned}$$

Rewriting in terms of x, this is

$$\tfrac{2}{3}(x^2 + 4)^{\frac{3}{2}} + c.$$

2772. (a) i. The accelerations of the masses are equal iff the strings are inextensible.

ii. The tension is the same throughout each string iff the pulleys are smooth and the strings are light.

(b) The result can be seen immediately, without any consideration of tensions, by considering NII along the strings. We can set up a single NII for the whole system because acceleration has the same magnitude for all objects.

The resultant force is

$$\begin{aligned} F &= m_3 g - m_1 g \\ &\equiv (m_3 - m_1)g. \end{aligned}$$

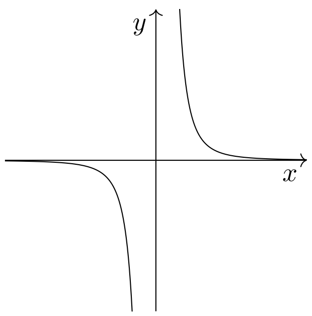
The total mass is

$$M = m_1 + m_2 + m_3.$$

Hence, the acceleration a ms^{-2} is given by

$$a = \frac{(m_3 - m_1)g}{m_1 + m_2 + m_3}.$$

2773. The index is odd, so $y = \frac{1}{x^3}$ has broadly the same shape as $y = \frac{1}{x}$:



2774. (a) Exponentiating both sides with base k, we get $x = k^y$. Then $k = e^{\ln k}$, so $x = e^{y \ln k}$.

(b) Differentiating with respect to y,

$$\begin{aligned} \frac{dx}{dy} &= \ln k \times e^{y \ln k} \equiv x \ln k \\ \implies \frac{dy}{dx} &= \frac{1}{x \ln k}. \end{aligned}$$

2775. The condition has no effect, as knowledge of the sum doesn't tell us anything about the order of the summands. There are $4! = 24$ different orders of 4 values. Hence, the probability is $\frac{1}{24}$.

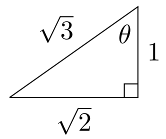
— Nota Bene —

A key clue here is the instruction "Write down...". This is exam-speak for "With (barely any or) no calculation, give the answer to...". Now, in this case, there is a tiny bit of calculation to do, in working out that 4! is 24. But the point is that the question, other than the instruction "Write down...", looks as if it might require a great deal of complicated calculation. And it doesn't!

2776. Translating the entire problem by vector $-k\mathbf{j}$, the reflections in $y = x$ are (a) $y = x$, (b) $x = y^2$, (c) $x = y^3$. Translating the entire problem back by vector $k\mathbf{j}$, the answers are

(a) $y = x + k$,
(b) $x = y^2 + k$,
(c) $x = y^3 + k$.

2777. The angle between planes is the angle between their normals. The normal to $\triangle ABC$, through its centroid, is a space diagonal through two vertices of the cube. Its length is $\sqrt{3}$. The normal through the base is vertical. So, the relevant triangle is



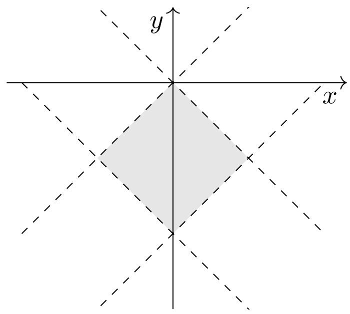
In the above, $\theta = \arctan \sqrt{2}$. Since this is the angle between the normals (space diagonal and vertical), it is also the angle between $\triangle ABC$ and the base.

2778. The one with the smallest mew (μ).

2779. The boundary equations are

$$\begin{aligned} x + y + 1 &= \pm 1, \\ x - y - 1 &= \pm 1. \end{aligned}$$

These are two pairs of parallel lines. The region which simultaneously satisfies both inequalities is therefore a square, with vertices at $(0, 0)$, $(1, 1)$, $(0, -2)$ and $(1, -1)$:



Its side length is $\sqrt{2}$, so its area is 2.

2780. The leading coefficient of both $\mathrm{f}(x)$ and $\mathrm{g}(x)$ is 1. Therefore, we can write $\mathrm{f}(x) = \mathrm{g}(x) + ax + b$, with a and b chosen to match the remaining coefficients (of x^1 and x^0). Splitting up the fraction,

$$\begin{aligned} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} &\equiv \frac{\mathrm{g}(x) + ax + b}{\mathrm{g}(x)} \\ &\equiv 1 + \frac{ax + b}{\mathrm{g}(x)} \\ &= 1 + \frac{ax + b}{x^2 + cx + d}, \text{ as required.} \end{aligned}$$

2781. Since the first, third and fifth terms occur every two terms, they must also be in GP. The common ratio is r^2, where r is the common ratio of the main GP. Equating two values of r^2,

$$\begin{aligned}\frac{d}{2d-3} &= \frac{2d+3}{d}\\ \Longrightarrow d &= \pm\sqrt{3}.\end{aligned}$$

These values give $r^2 = \frac{d}{2d-3} = 2 \pm \sqrt{3}$. Both of these are positive, so are answers to the problem.

Nota Bene

If either of the answers for r^2 had been negative, then we would have rejected it: a negative value for r^2 would have produced no real value for r.

2782. (a) The second derivative is

$$\mathbf{a} = \begin{pmatrix} 6t \\ -12t \\ 6t \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ 6 \end{pmatrix} t \text{ ms}^{-2}.$$

Since $\mathbf{a} = \mathbf{k}t$, acceleration increases linearly with time.

(b) The first derivative is

$$\mathbf{v} = \begin{pmatrix} 3t^2 \\ -6t^2 \\ 3t^2 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} t^2 \text{ ms}^{-1}.$$

Since $t^2 \geq 0$, this never changes direction.

2783. The sum, for $n > 2$, is as follows:

$$\begin{aligned}& n^3 + (n+1)^3 + (n+2)^3 \\ \equiv\ & 3n^3 + 9n^2 + 15n + 9 \\ \equiv\ & 3(n+1)(n^2+2n+3).\end{aligned}$$

Since $n > 2$, the factors 3, $(n+1)$ and (n^2+2n+3) are distinct, because they are in strictly ascending order. So, the sum has 3, $(n+1)$ and (n^2+2n+3) as factors, as well as 1 and itself. Therefore, it has at least five factors. QED.

2784. (a) With $X \sim B(6, 0.5)$, $\mathbb{P}(X = 3) = \frac{5}{16}$.

(b) We are looking for orders of UUUDDD (wrapped in a circle) in which each letter appears in a two or a three.

In a successful outcome, UU must appear. If it is surrounded by two Ds, forming DUUD, then the other U must be on its own. Hence, the three Us (and three Ds) must appear together. There are 6 ways of doing this, out of ${}^6\text{C}_3 = 20$ total outcomes. This gives $p = \frac{6}{20} = \frac{3}{10}$.

2785. Setting the output to y, and assuming that the domain and codomain are suitably restricted,

$$\begin{aligned} y &= \frac{\sin x + 1}{\sin x - 1} \\ \Longrightarrow y\sin x - y &= \sin x + 1 \\ \Longrightarrow \sin x(y-1) &= y+1 \\ \Longrightarrow \sin x &= \frac{y+1}{y-1} \\ \Longrightarrow x &= \arcsin\left(\frac{y+1}{y-1}\right).\end{aligned}$$

So, the inverse is $\text{f}^{-1} : x \mapsto \arcsin\left(\frac{x+1}{x-1}\right)$.

2786. If $y = \text{f}(x)$ has rotational symmetry around the origin, then, for all x,

$$\text{f}(-x) = -\text{f}(x).$$

Differentiating by the chain rule,

$$\begin{aligned} -\text{f}'(-x) &= -\text{f}'(x) \\ \Longrightarrow \text{f}'(-x) &= \text{f}'(x).\end{aligned}$$

This is true for all x, so $y = \text{f}'(x)$ has the y axis as a line of symmetry. □

Nota Bene

In the language of odd and even functions:

- the graph of an odd function has rotational symmetry around the origin,
- the graph of an even function has the y axis as a line of symmetry.

In this question, the derivative of an odd function is even. It is also true that the derivative of an even function is odd.

2787. Solving the boundary equation,

$$\begin{aligned} X^2 + X &= 10 \\ \Longrightarrow X &= \frac{-1 \pm \sqrt{41}}{2}.\end{aligned}$$

The solution to the inequality $X^2 + X > 10$ is all X values outside these roots. So, we calculate

$$\begin{aligned}\mathbb{P}\left(X < \tfrac{-1-\sqrt{41}}{2}\right) &= 0.001862... \\ \mathbb{P}\left(X > \tfrac{-1+\sqrt{41}}{2}\right) &= 0.778205...\end{aligned}$$

Adding these,

$$\begin{aligned}\mathbb{P}(X^2 + X > 10) &= 0.001862 + 0.778205 \\ &= 0.780 \text{ (3sf)}.\end{aligned}$$

2788. (a) The area is $A = xy$. Differentiating by the product rule,

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}.$$

Substituting in,

$$\begin{aligned}\frac{dA}{dt} &= 20 \times 360 + 240 \times 50 \\ &= 19200 \text{ pixels per second.}\end{aligned}$$

(b) A "pixel" is a square. It can be used both as a unit of area, as in (a), but also as a unit of length, as in the original question. This is as opposed to the centimetre, where different units cm and cm^2 are used for length and area.

2789. We rearrange to $3x^3 + 2x^2 + 20x - 7 = 0$. Then, using either a polynomial solver or a numerical method such as N-R, we find the root $x = \frac{1}{3}$. So, $(3x - 1)$ is a factor. Taking it out,

$$(3x - 1)(x^2 + x + 7) = 0.$$

The quadratic factor has $\Delta = -27 < 0$, so there is exactly one real root, $x = \frac{1}{3}$.

2790. A counterexample is $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, $\mathbf{c} = \mathbf{k}$, and $\mathbf{d} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$. The sum is zero. But, since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in the x, y, z directions, there is no single plane containing all three of them.

Nota Bene

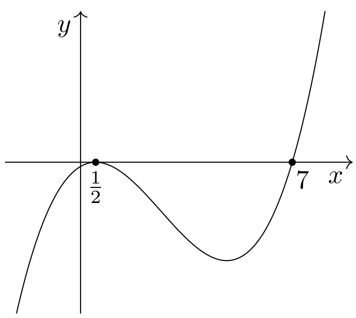
Note that there *is* a single plane ($x + y + z = 1$) which contains the points with *position* vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. That's the difference between the point $(1, 0, 0)$ and the vector $\mathbf{i}$. They are not the same mathematical object: the point $(1, 0, 0)$ lies in the plane $x + y + z = 1$, but the vector $\mathbf{i}$ doesn't. The vector $\mathbf{i}$ *gets you onto the plane from the origin.*

2791. (a) There is a common factor of $(2x - 1)^2$. Since this is a square, there is a point of tangency with the x axis at the associated root. This is a stationary point.

(b) The SP lies at $(1/2, 0)$.

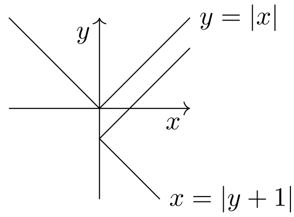
(c) Using a calculator, the other root lies at $x = 7$.

(d) The graph is a positive cubic with a double root at $x = \frac{1}{2}$ and a single root at $x = 7$:

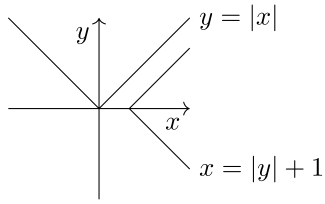


2792. (a) They do intersect. The line $y = x$, for $x, y \geq 0$, is common to both graphs.

(b) They do not intersect:



(c) They do not intersect:



2793. (a) At each step, the parity i.e. the odd/evenness changes. We begin at 0, which is even, at $t = 0$. Hence, after an odd number of steps, five in this case, we must end up at an odd number.

(b) To end up at -3, we must get, in some order, $+----$. The number of orders of these is 5, which gives a probability $p = \frac{5}{32}$.

2794. (a) K_1 is a triangle with 3 line segments. In each successive version, the number of line segments is quadrupled. This produces a GP, with nth term $3 \times 4^{n-1}$.

(b) K_n has $3 \times 4^{n-1}$ line segments. So, since a new triangle is added to each line segment, $3 \times 4^{n-1}$ is also the number of triangles added to form K_{n+1}. Each of these has area $(1/9)^n$. The total area added is therefore

$$\begin{aligned}&3 \times 4^{n-1} \times (1/9)^n \\ \Longrightarrow\ &3/4 \times (4/9)^n.\end{aligned}$$

(c) The total area added is

$$\tfrac{3}{4} \times \left(\tfrac{4}{9}\right)^1 + \tfrac{3}{4} \times \left(\tfrac{4}{9}\right)^2 + ...$$

This is a geometric series with first term $a = \frac{1}{3}$ and common ratio $\frac{4}{9}$. Its sum to infinity is

$$\begin{aligned}S_\infty &= \frac{a}{1 - r} \\ &= \frac{\frac{1}{3}}{1 - \frac{4}{9}} \\ &= 0.6.\end{aligned}$$

The limit is $1+0.6 = 160\%$ of the original area.

2795. Reframe the problem with $t = 0$ at the moment the first cone is passed. With initial velocity u and constant acceleration a, the first 5 metres gives

$$5 = u(0.18) + \tfrac{1}{2}a(0.18)^2.$$

The first 10 metres gives

$$10 = u(0.365) + \tfrac{1}{2}a(0.365)^2.$$

Solving simultaneously, $u = 28.148...$ ms^{-1} and $a = -4.1137...$ ms^{-2}. Using these in $v^2 = u^2 + 2as$, the total distance travelled is

$$\begin{aligned} s &= -\frac{u^2}{2a} \\ &= -\frac{28.148^2}{2 \cdot -4.1137} \\ &\approx 96.3. \end{aligned}$$

Dividing by 5, the car passes 19 cones.

2796. Setting up $\text{f}(x) = x^3 - x^2 - 63840000$, we want to solve $\text{f}(x) = 0$. The N-R iteration is

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 63840000}{3x_n^2 - 2x_n}.$$

Beginning with $x_0 = 100$ (or virtually any other initial guess), $x_n \to 400$.

— Nota Bene —

Newton-Raphson is the numerical method to go for in most cases at A-level, if the choice is yours. It is much more reliable than fixed-point iteration, which regularly fails to find what you're looking for, and much less work than decimal search.

2797. Using log rules, we can simplify the derivative:

$$\begin{aligned} &\frac{\ln a^x + \ln a}{\ln a^x - \ln a} \\ \equiv\ &\frac{x \ln a + \ln a}{x \ln a - \ln a} \\ \equiv\ &\frac{x+1}{x-1}. \end{aligned}$$

This is independent of a. We can now rewrite as a proper algebraic fraction, and then integrate:

$$\begin{aligned} y &= \int 1 + \frac{2}{x-1}\, dx \\ &= x + 2\ln|x-1| + c. \end{aligned}$$

2798. Both statements are true. (b) is true, because we can generate every value in $[0, 4]$ as the square of one in $[0, 2]$. And (a) is true for the same reason; it is implied by (b).

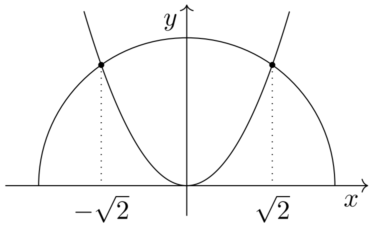
2799. Vertically, $y = 3 + 1.4t - 4.9t^2$. Horizontally, $x = 0.7t$. This gives $t = \frac{10}{7}x$. Substituting into the vertical equation,

$$\begin{aligned} & y = 3 + \tfrac{14}{7}x - \tfrac{490}{49}x^2 \\ \Longrightarrow\ & y = 3 + 2x - 10x^2. \end{aligned}$$

2800. Setting up the boundary equation,

$$\begin{aligned} & \sqrt{6 - x^2} = x^2 \\ \Longrightarrow\ & 6 - x^2 = x^4 \\ \Longrightarrow\ & x^4 + x^2 - 6 = 0 \\ \Longrightarrow\ & (x^2+3)(x^2-2) = 0. \end{aligned}$$

The first factor has no real roots, so the boundary equation has solution $x = \pm\sqrt{2}$. Sketching the LHS and RHS:



We need the x values for which the semicircle is strictly above the parabola, so $x \in (-\sqrt{2}, \sqrt{2})$.

— End of 28th Hundred —